HYDRODYNAMIC STABILITY OF A GENERALIZED NONLINEAR VISCOPLASTIC FLUID UNDER PLANE GRADIENT FLOW

S. L. Simkhovich, A. S. Romanov, and L. I. Ionochkina

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The hydrodynamic stability of a generalized nonlinear viscoplastic fluid under plane gradient flow is studied in the linearized theory.

The hydromechanics of fluids with complex rheological properties are of great interest in modern engineering processes. The mechanical properties of many real fluids can be described by a generalized nonlinear viscoplastic fluid model of the form [1]:

$$\tau_{ij} = 2\left[\frac{\tau_0^{1/n}}{\Omega^{1/m}} + \mu^{1/m}\right]^n \Omega^{\frac{n}{m}-1} f_{ij}. \tag{1}$$

Here $\Omega = \sqrt{2f_{ij}f_{ij}}$ is the magnitude of the deformation rate tensor given by $f_{ij} = (1/2)\cdot(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$.

Below we study the hydrodynamic stability for stationary gradient flow of the rheologically complex medium (1) in a plane channel against infinitestimal two-dimensional perturbations.

Let the fluid (1) move in a plane channel under a longitudinal pressure gradient $\partial P/\partial x = \text{const} < 0$. Because of the presence of the yield stress τ_0 in the rheological equation (1), under stationary flow a quasirigid region will be formed in the center of the channel where the deformation vanishes: $\Omega \equiv 0$. In the viscoplastic flow region near the edge of the channel, the velocity of motion of the fluid $U = U(y)$ for the lower half of the channel will be given by

$$U = \begin{cases} \dfrac{n+m}{n} \displaystyle\int_{-1}^{y} [(-y)^{1/n} - \xi^{1/n}]^m \, dy, & -1 \leqslant y \leqslant -\xi; \\ U(-\xi) = \text{const}, & -\xi \leqslant y \leqslant 0, \end{cases} \tag{2}$$

where ξ is the dimensionless half-width of the quasirigid region. This is determined from the equilibrium condition

$$\xi = \varkappa \left[\frac{n}{n+m}\right]^{n/m}, \quad \varkappa = \tau_0 [L/\mu U_0]^{n/m}.$$

We introduce the dimensionless stream function $\psi = \psi(x, y, t)$ satisfying $u_x = \partial\psi/\partial y$, $u_y = -\partial\psi/\partial x$. Then ψ will be given by

$$\psi(x, y, t) = \int_0^y U dy + \varphi(y) \exp[i\alpha(x - ct)],$$

where α and αc are the wave number and complex frequency of the perturbation, respectively.

Linearizing the equation of motion for the perturbation with respect to the amplitude $\varphi = \varphi(y)$ of the perturbation as in [2], we obtain the ordinary differential equation

$$(U - c)(\varphi'' - \alpha^2\varphi) - U''\varphi = \frac{1}{i\alpha \operatorname{Re}} \left\{(\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi)\eta_0 + \right.$$

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$$+ 2\eta_0'(\varphi'' - \alpha^2\varphi') + \eta_0''(\varphi'' + \alpha^2\varphi) + \left[u'\left(\frac{d\eta}{d\omega}\right)_0(\varphi'' + \alpha^2\varphi)\right]'' + \alpha^2 U'\left(\frac{d\eta}{d\omega}\right)_0(\varphi'' + \alpha^2\varphi)\Big\}, \qquad (3)$$

which is valid in the viscoplastic region as well as in the quasirigid core. The subscript 0 denotes the unperturbed flow and a prime means differentiation with respect to y. $\eta = \eta(\omega)$ is a dimensionless apparent viscosity which is a function of the magnitude of the deformation rate tensor Ω [3]. Finally, $\omega = \Omega L/U_0$; $\mathrm{Re} = \rho L^{n/m} U_0^{2-m/n}/\mu^{n/m}$ is the generalized Reynolds number.

In the viscoplastic flow region $\eta = [1 + \varkappa^{1/n}/\omega^{1/m}]^n$, $-1 \leqslant y \leqslant -\xi$ and (3) is supplemented by the boundary conditions [2]

$$\varphi'(-1) = \varphi(-1) = \varphi'''(0) = \varphi'(0) = 0. \qquad (4)$$

According to the theory of asymptotic expansions of ordinary differential equations, the four linearly independent solutions of (3) in the viscoplastic flow region $-1 \leqslant y \leqslant -\xi$ have the form [4, 5]

$$\varphi_1 = \sum_{k=0}^{\infty} \alpha_k (y - y_c)^{k+1}; \quad \varphi_2 = \alpha_1\varphi_1 \ln(y - y_c) + \sum_{k=0}^{\infty} b_k (y - y_c)^k;$$

$$\varphi_{3,4} = \int_{\pm\infty}^{z} dz \int_{\pm\infty}^{z} z^{1/2} H_{1/3}^{(1,2)}\left[\frac{2}{3}(iz)^{3/2}\right] dz + O[(\alpha\,\mathrm{Re})^{-1/3}],$$

where $z = (y - y_c)/(\alpha\,\mathrm{Re}\,U'(y_c))^{1/3})$ and the coefficients α_k, b_k are given by recurrence relations [5].

In the core region $-\xi \leqslant y \leqslant 0$ the corresponding solution is [5, 6]:

$$\varphi_{1,2}^* = \exp(\pm\alpha y), \qquad \varphi_{3,4}^* = \exp\left\{\pm\sqrt{i\alpha\,\mathrm{Re}} \int_0^y \left[(U - c)/\left(\eta_0 + U'\left(\frac{d\eta}{d\omega}\right)_0\right)\right]^{1/2} dy\right\} \{1 + O[(\alpha\,\mathrm{Re})^{-1/2}]\}.$$

The general solution of (3) is then given by

$$\varphi = \sum_{k=1}^{4} c_k\varphi_k, \quad -1 \leqslant y \leqslant -\xi;$$

$$\varphi^* = \sum_{k=1}^{4} c_k\varphi_k^*, \quad -\xi \leqslant y \leqslant 0. \qquad (5)$$

The solution (5) must satisfy the boundary conditions (4) and the matching condition

$$\frac{d^k\varphi}{dy^k}(-\xi) = \frac{d^k\varphi^*}{dy^k}(-\xi), \quad k = 0, 1, 2, 3, \qquad (6)$$

which to order $O(\alpha\mathrm{Re})^{-1/3}$ leads to the secular equation:

$$\frac{\varphi_3(-1)}{\varphi_3'(-1)} = \frac{\begin{vmatrix} \varphi_2(-1) & \varphi_1(-1) \\ \varphi_2'(-\xi) + \alpha\varphi_2(-\xi)\,\mathrm{th}\,\alpha\xi & \varphi_1'(-\xi) + \alpha\varphi_1(-\xi)\,\mathrm{th}\,\alpha\xi \end{vmatrix}}{\begin{vmatrix} \varphi_2'(-1) & \varphi_1'(-1) \\ \varphi_2'(-\xi) + \alpha\varphi_2(-\xi)\,\mathrm{th}\,\alpha\xi & \varphi_1'(-\xi) + \alpha\varphi_1(-\xi)\,\mathrm{th}\,\alpha\xi \end{vmatrix}}. \qquad (7)$$

Thus, the study of stability of the fluid (1) reduces to finding the eigenvalues of the secular equation (7).

In Fig. 1 we show neutral stability curves in the (α, Re) plane, constructed from the numerical solution of (7) for different values of m and n and different sizes of the quasi-rigid zone ξ.

The dependence of the critical Reynolds number Re_{cr} on the rheological constants m, n and the width ξ is shown in Figs. 2-4. Clearly, an increase in the rheological constant m ($n > 1$) increases the value of the critical Reynolds number Re_{cr} at which loss of stability occurs. An increase in the rheological constant n at constant m destabilizes the flow (Fig. 3).

The effect of the rheological parameters of the medium on the flow stability can be explained by the effect of Reynolds stress on the transfer of energy from the unperturbed flow

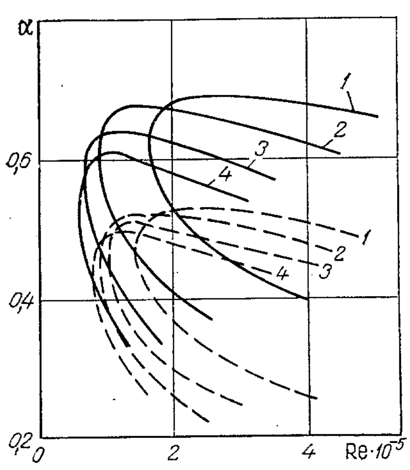

Fig. 1

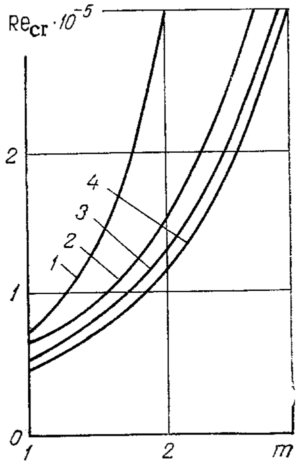

Fig. 2

Fig. 1. Neutral stability curves for m = 0.8 (ξ = 0.4 for the solid curves, ξ = 0.6 for the dashed curves): 1) n = 0.4; 2) 0.6; 3) 0.8; 4) 1.

Fig. 2. Dependence of Re_{cr} on m for ξ = 0.4: 1) n = 1; 2) 2; 3) 2.5; 4) 3.

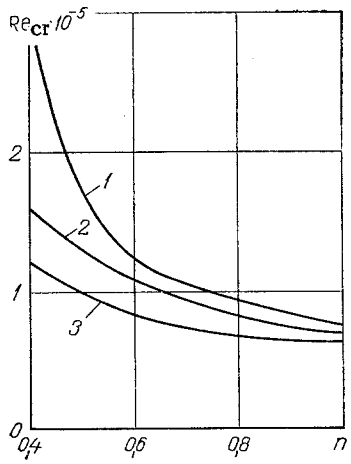

Fig. 3

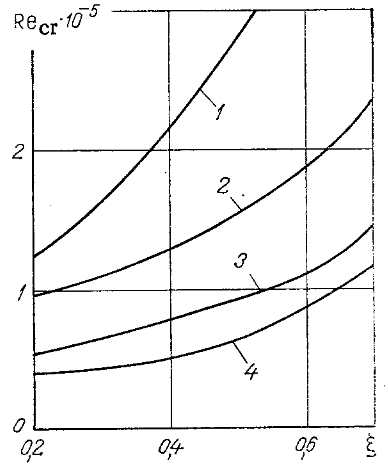

Fig. 4

Fig. 3. Dependence of Re_{cr} on n for ξ = 0.4: 1) m = 1; 2) 0.8; 3) 0.6.

Fig. 4. Dependence of Re_{cr} on ξ for n = 2.5: 1) m = 2.5; 2) 2; 3) 1.5; 4) 1.

to the perturbation. It is known [2] that a distortion of the velocity profile of the unperturbed flow at the critical point, given by the ratio of the derivatives $\frac{d^2U(y_c)}{dy^2}\left(\frac{dU(y_c)}{dy}\right)^{-1}$, strongly affects the Reynolds stress.

In our model the dependence of the Reynolds stress on the rheological parameters has the form

$$\tau \sim \left(\frac{d^2U(y_c)}{dy^2}\right)\left(\frac{dU(y_c)}{dy}\right)^{-1} \sim \frac{m}{n}\,\frac{(-y)^{\frac{1-n}{n}}}{\left[(-y)^{\frac{1}{n}} - \xi^{\frac{1}{n}}\right]}.$$

This relation can also be used to interpret the dependence of the critical Reynolds number on the rheological constants m and n and the half-width of the quasirigid region ξ observed in Figs. 2-4. The quasirigid region significantly affects the growth of perturbations in the flow. An increase in the half-width of the quasirigid region ξ stabilizes the flow (Fig. 4).

The rheological model (1) is a generalization of several well-known models. For example, if the yield stress τ_0 vanishes, Eq. (1) reduces to a rheological power-law equation. When $m = n$ ($\tau_0 = 0$) we obtain an ordinary Newtonian fluid. The value of Re_{cr} for $\tau_0 = 0$ and $m = n$ reduces to the value for a Newtonian fluid [2] (Fig. 4). Our results for Re_{cr} in the case $\tau = 0$ (Fig. 4) reduce to those for a non-Newtonian fluid with a rheological power law [6].

NOTATION

u_i, velocity vector components; τ_0, yield stress; μ, shear viscosity; m, n, rheological constants; U, velocity of the unperturbed flow; α, wave number; L, channel width; ρ, density; φ, perturbation amplitude; c, phase velocity; U_0, characteristic velocity; $H_{1/3}^{(1,2)}$, Hankel functions of the first and second kind of order 1/3; τ, Reynolds stress.

LITERATURE CITED

1. Z. P. Shul'man, Convective Heat and Mass Transfer of Rheologically Complex Fluids [in Russian], Énergiya, Moscow (1975).
2. Tsia Tsao Lin, Theory of Hydrodynamic Stability [Russian translation], IL, Moscow (1958).
3. N. V. Mikhailov and P. A. Rebinder, "On the structural mechanical properties of dispersive and high-molecular systems," Kolloidn. Zh., No. 17, 107-119 (1955).
4. W. Wasow, Asymptotic Expansions for Ordinary Differential Equations, Kreiger (1976).
5. L. K. Martinson, K. B. Pavlov, and S. L. Simkhovich, "Hydrodynamic stability of the gradient flow of a conducting fluid with a rheological power law in a transverse magnetic field," Magnitn. Gidrodin., No. 2, 35-40 (1972).
6. A. M. Makarov, L. K. Martinson, and K. B. Pavlov, "Stability of a non-Newtonian fluid with a rheological power law under plane flow," Inzh.-Fiz. Zh., 16, 793 (1969).

HYDRODYNAMIC INSTABILITY OF THE AXISYMMETRIC FLOW OF AN IDEAL FLUID WITH AN INTERPHASE

V. E. Epikhin and V. Ya. Shkadov

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We study the instability under simultaneous rotational and translational flow of a fluid and ambient medium in the cases of a cylindrical annular jet, capillary jet, and cylindrical fluid layers on the inner and outer surfaces of a cylinder.

The type of flow under study is schematically illustrated in Fig. 1. Reviews of the literature and some new experimental results on instabilities of jets can be found in [1-9]. The stability of a fluid on a rotating cylindrical surface was studied in [10, 11]. In the present paper the stability of potential flow is considered in the most general formulation. Such flows are used in vaporizers, heat-transfer devices, chemical reactors, in the paper-pulp industry, and also in vertical—centrifugal methods of producing mineral fibers [11].

In a cylindrical coordinate system with the axis of coordinates taken along the symmetry axis of the problem, the flow is described by the potential functions

$$\Phi_f^0(X, \theta) = U_f X + \Gamma_f \theta, \quad \Phi_{e,i}^0 = U_{e,i} X + \Gamma_{e,i}\theta. \tag{1}$$

where the second term in both equations gives the velocity potentials of line vortices along the axis of rotation with circulations $2\pi\Gamma_f$, $2\pi\Gamma_e$, $2\pi\Gamma_i$, respectively [1]. At $t = 0$ a potential wave perturbation of infinitesimal amplitude is applied to the unperturbed flow. The potential functions of nonsteady motion satisfy the Laplace equation and the Cauchy—Lagrange integral in a flow region to be determined as part of the solution. The boundary conditions express the jump in the normal stress due to surface tension, the continuity of streamline flow at the boundaries, and the boundedness of the potentials on the axis and at infinity, and also the periodicity

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